Trigonometric Integrals

Compute $I = \int_0^{2\pi} U(\cos\theta, \sin\theta)\,d\theta$, where U is a rational function of $\cos\theta$ and $\sin\theta$. To compute I, let C be the unit circle with center 0 and oriented once in the ~~[illegible]~~ counterclockwise direction. Parametrize C as $z = e^{i\theta}$, $0 \le \theta \le 2\pi$. Then

$$\begin{cases} \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{1}{2}\left(z + \dfrac{1}{z}\right), \\ \sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{1}{2i}\left(z - \dfrac{1}{z}\right), \\ dz = ie^{i\theta}d\theta \implies d\theta = \dfrac{1}{iz}dz. \end{cases}$$

So $I = \int_C U\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right)\frac{1}{iz}dz$, a contour integral.

Example (Exercise 3 in Chapter 14)

Let $a \in \mathbb{C}$ with $|a| < 1$. Compute

$$I = \int_0^{2\pi} \frac{1}{1 - 2a\cos\theta + a^2}\,d\theta.$$

Solution:

$$I = \int_C \frac{1}{1 - \cancel{2}a\left(z + \frac{1}{z}\right)\frac{1}{\cancel{2}} + a^2}\,\frac{1}{iz}\,dz$$

$$= \frac{1}{i}\int_C \frac{1}{z - a(z^2+1) + a^2 z}\,dz$$

$$= -\frac{1}{i}\int_C \frac{1}{a(z^2+1) - (1+a^2)z}\,dz = -\frac{1}{i}\int_C \frac{1}{az^2 - (1+a^2)z + a}\,dz$$

Factorizing the denominator

$$z = \frac{(1+a^2) \pm \sqrt{(1+a^2)^2 - 4a^2}}{2a} = \frac{(1+a^2) \pm \sqrt{1 + 2a^2 + a^4 - 4a^2}}{2a}$$

$$= \frac{(1+a^2) \pm \sqrt{1 - 2a^2 + a^4}}{2a} = \frac{(1+a^2) \pm \sqrt{(1-a^2)^2}}{2a}$$

$\therefore z = \dfrac{1+a^2 \pm (1-a^2)}{2a} = \dfrac{1}{a}$ or a. We take a only because a is inside C. So,

$$I = \frac{1}{i}\int_C \frac{1}{a(z-a)(z-\frac{1}{a})}\,dz$$

Let $f(z) = \dfrac{1}{a(z-a)(z-\frac{1}{a})}$. Then a is a simple pole of f.

By Cauchy's Residue Theorem,

$$\begin{aligned} I &= -\frac{1}{i}\,2\pi i\,\mathrm{Res}(f,a) \\ &= -\frac{1}{i}\,2\pi i \lim_{z\to a}(z-a)f(z) = -2\pi \lim_{z\to a}\frac{1}{(z-\frac{1}{a})a} \\ &= -2\pi\frac{a}{(a^2-1)a} = 2\pi\frac{\cancel{a}\,1}{1-a^2} \end{aligned}$$

$$\therefore I = \frac{2\pi}{1-a^2}.$$

Cauchy's Principal Values of Improper Integrals on $(-\infty,\infty)$

Let f be a continuous complex-valued function on $[0,\infty)$.

Definition: The Cauchy Principal Value $pv\int_{-\infty}^{\infty} f(x)\,dx$ of the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$pv\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\rho\to\infty}\int_{-\rho}^{\rho} f(x)\,dx,$$

if the limit exists.

Fourier Transforms

Let f be a continuous complex-valued function on $(-\infty,\infty)$. Then we define the Fourier transform $\hat{f}$ of f on $(-\infty,\infty)$ by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}}\, pv\int_{-\infty}^{\infty} e^{-ix\xi} f(x)\,dx,\quad \xi\in(-\infty,\infty)$$

if the Cauchy principal value exists.

Sometimes we omit pv with no confusion.

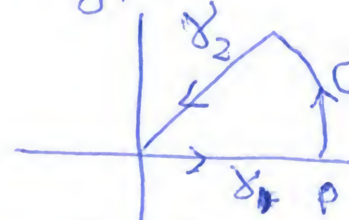
Closely related to the Fourier transform are the cosine transform and sine transform, respectively given by

$$\frac{1}{\sqrt{2\pi}}\,pv\int_{-\infty}^{\infty}\cos(x\xi)f(x)\,dx \quad\text{and}\quad \frac{1}{\sqrt{2\pi}}\,pv\int_{-\infty}^{\infty}\sin(x\xi)f(x)\,dx.$$

Example: Compute $\int_0^{\infty}\cos(x^2)\,dx$ and $\int_{-\infty}^{\infty}\sin(x^2)\,dx$.
(Fresnel Integrals)
(Ex 4 in Chapter 16)

Solution: We first compute $\int_C e^{iz^2}dz$, where C is the contour given by

$$C=\gamma_1+C_\rho+\gamma_2,$$



so by Cauchy's integral theorem,

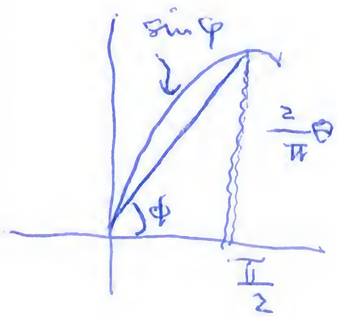
$$0=\int_C e^{iz^2}dz=\int_0^{\rho}e^{ix^2}dx+\int_0^{\frac{\pi}{4}}e^{i\rho^2e^{2i\theta}}\,i\rho e^{i\theta}\,d\theta - e^{i\frac{\pi}{4}}\int_0^{\rho}e^{-r^2}dr.$$

Now,

$$\left|\int_0^{\pi/4}e^{i\rho^2e^{2i\theta}}\,i\rho e^{i\theta}\,d\theta\right|\le \rho\int_0^{\pi/4}e^{-\rho^2\sin 2\theta}\,d\theta$$

$$\le \rho\int_0^{\frac{\pi}{4}}e^{-\rho^2 4\theta/\pi}\,d\theta$$

$$=\rho\left.\frac{-e^{-\frac{\rho^2 4}{\pi}\theta}}{\frac{\rho^2 4}{\pi}}\right|_0^{\frac{\pi}{4}}$$



$$=\rho\left(1-e^{-\rho^2}\right)\Big/\left(\frac{\rho^2 4}{\pi}\right)$$

$$=\frac{\pi}{4\rho}\left(1-e^{-\rho^2}\right)\to 0$$

so $\int_0^{\infty}e^{ix^2}dx=e^{i\frac{\pi}{4}}\int_0^{\infty}e^{-x^2}dx$ as $\rho\to\infty$.

$$\therefore \int_0^{\infty} e^{i x^2}\,dx = e^{i\pi/4}\sqrt{\pi}$$

$$= \left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right)\sqrt{\pi}$$

$$\therefore \int_0^{\infty} \cos(x^2)\,dx = \frac{\sqrt{2}}{2}\sqrt{\pi} = \sqrt{\frac{\pi}{2}}$$

$$\int_0^{\infty} \sin(x^2)\,dx = \sqrt{\frac{\pi}{2}}$$